

Localized bases in $L^2(0, 1)$ and their use in the analysis of Brownian motion

Palle E. T. Jorgensen ^{*,1}

Department of Mathematics, The University of Iowa, 14 MacLean Hall, Iowa City, IA 52242-1419, U.S.A.

Anilesh Mohari

S. N. Bose Centre for Basic Sciences, JD Block, Sector-3, Calcutta-98, India

Abstract

Motivated by problems on Brownian motion, we introduce a recursive scheme for a basis construction in the Hilbert space $L^2(0, 1)$ which is analogous to that of Haar and Walsh. More generally, we find a new decomposition theory for the Hilbert space of square-integrable functions on the unit-interval, both with respect to Lebesgue measure, and also with respect to a wider class of self-similar measures μ . That is, we consider recursive and orthogonal decompositions for the Hilbert space $L^2(\mu)$ where μ is some self-similar measure on $[0, 1]$. Up to two specific reflection symmetries, our scheme produces infinite families of orthonormal bases in $L^2(0, 1)$. Our approach is as versatile as the more traditional spline constructions. But while singly generated spline bases typically do not produce orthonormal bases, each of our present algorithms does.

Key words: Haar, Walsh, orthonormal basis, Hilbert space, Cuntz relations, irreducible representation, wavelets, iterated function system, Cantor set

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^{*} Tel: (319) 335-0782, Fax: (319) 335-0627, E-mail: jorgen@math.uiowa.edu

Email addresses: jorgen@math.uiowa.edu (Palle E. T. Jorgensen),
anilesh@boson.bose.res.in (Anilesh Mohari).

URL: <http://www.math.uiowa.edu/~jorgen> (Palle E. T. Jorgensen).

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1 Introduction

The basis constructions considered in this paper involve elements from the theory of operator algebras. Since this may not be widely known to readers in approximation theory, we begin with a few remarks.

The study of operator algebras breaks up in two parts: One the study of “the algebras themselves” as they emerge from the axioms, von Neumann algebras, and C^* -algebras. The other has a more applied slant: It involves “representations” of the algebras. There is a close connection between the two parts of the theory: For example, representations of C^* -algebras generate von Neumann algebras. It was realized in the last ten years (see e.g., [BJP96], [DuJo05]) that certain families of representations are useful in basis constructions in harmonic analysis, in approximations, in signal/image analysis, and more generally in computational mathematics. The bases in question may typically be built up from representations of an especially important family of simple C^* -algebras, known as the Cuntz algebras [Cu77]. These Cuntz algebras (see Lemma 2.2 below) are denoted $\mathcal{O}_2, \mathcal{O}_3, \dots$, including \mathcal{O}_∞ .

The connection to Cuntz algebras \mathcal{O}_N is further relevant to the kind of dynamical systems built on iterated branching-laws, with the case of \mathcal{O}_N representing N -fold branching. The reason for this is that if N is fixed, \mathcal{O}_N includes in its definition an iterated branching, taking the form of subdivision, but now within the context of Hilbert space; so we generate subdivisions into orthogonal families of subspaces of the initial Hilbert space. In this paper, we follow up on a certain probabilistic aspect of this construction.

The \mathcal{O}_N point of view is especially well suited to basis constructions in such contexts as wavelets and fractals since they naturally involve the same kind of sub-division. Our starting point is an initial Hilbert space \mathcal{H} , where \mathcal{H} may be $L^2(\mathbb{R}^d)$, or \mathcal{H} may be $L^2(\mu)$ for some fractal measure μ ; see Section 2 below. The more successful bases in Hilbert space are the orthonormal bases ONBs, and we shall consider a certain computational algorithm for generating them.

A further reason the subdivision schemes in Hilbert space are useful is that the more familiar Fourier wave functions are periodic, and so not localized. Moreover these existing Fourier tools are typically not friendly to algorithmic computations. By a *local* (See Definition 1.1) construction we mean an algorithm which begins with a finite family of functions (often one or two) having a fixed compact (i.e., local) support, and a procedure allowing assigned scaling and translation operations. As is popular in the sub-band approach to wavelets and wavelet packets in 1D, the scaling is typically in powers of a fixed base, i.e., it could consist of powers N^j where N is fixed ($N > 1$) and where j varies over \mathbb{Z} , stretching and squeezing the support.

The main result in this paper concerns bases in the Hilbert spaces $L^2(X, \mu)$ defined from measures μ arising as equilibrium measures (also called self-similar measures) for iterated function systems (IFS). However, our results are spelled out in more detail for $L^2((0, 1); \text{Lebesgue})$, where the classical Walsh system is a special case. Our construction uses ideas from dynamical systems and operator algebras (specifically representations of the Cuntz algebras). The Cuntz algebra [Cu77] is used in the construction of bases for geometric structures with self-similarity such as iterated function systems (IFS). Recall that for a fixed finite N , the Cuntz algebra \mathcal{O}_N is generated abstractly by N isometries. In representations of \mathcal{O}_N on a concrete Hilbert space \mathcal{H} , the resulting isometries S_i , say, have orthogonal ranges which form a partition of unity in the particular Hilbert space \mathcal{H} which carries the representation, in the sense that the identity operator $\mathbb{1}_{\mathcal{H}}$ is written as a sum of the N projections $S_i S_i^*$ onto the respective ranges $S_i \mathcal{H}$. Since the subdivision process can be iterated, this idea has already proved useful in understanding orthogonal families in Hilbert spaces built on IFSs; see, e.g., [DuJo06b, JoPe96, JoPe98]. Our analysis here uses such particular \mathcal{O}_N representations in combination with certain graph-theoretic considerations. In addition to the reference [Cu77], the Cuntz algebras \mathcal{O}_N and their representations are reviewed in [Jor06, Sections 7.6 and 7.7]. The book [Jor06] also includes additional motivation, and [Jor06, Chapter 4, p. 69, and Section 9.4] cover details for an IFS-family of Cantor systems and their self-similar measures.

Applications to Brownian motions are intended but postponed to a later paper. The connection between Brownian motion, Cuntz algebras, and IFSs is treated in the literature, for example in [Jor06, pp. 56–57, 151, and 203].

There has been a recent increased interest in basis constructions outside the traditional setting of harmonic analysis. The setting which so far has proved more amenable to an explicit analysis with basis functions involves a mix of analysis and dynamics, and it typically goes beyond the standard and more familiar setting of orthonormal bases consisting of Fourier frequencies. The context of frames in Hilbert space (see, e.g., [ALTW04], [BoPa05], and [BPS03]) is a case in point.

As is well known, the classical setting of Fourier analysis presupposes a choice of Fourier frequencies, or Fourier trigonometric basis functions. However, this unduly limits our choices, and the applications: As is well known, Fourier's basis functions are less localized, and the computational formulas typically are not recursive. There are now alternative dynamical approaches which are recursive, and at the same time are amenable to harmonic basis constructions; see, e.g., [Dut05], [JoPe98], [DuJo05], and [DuJo06a]. Moreover, these recursive models arise in applications exhibiting a suitable *scale-similarity*. Their consideration combines classical ideas from infinite convolution with ideas from dynamics of a more recent vintage.

In this paper, we introduce a recursive scheme for a basis construction in the Hilbert space $L^2(0, 1)$ which is analogous to that of Haar [Haa10] and Walsh [Wal23, Chr55]. While computationally efficient, these more traditional approaches limit the choices of functions too much, often to step functions, or at any rate to functions that have a limited number of derivatives.

We begin here with a certain dual system of axioms for reflection symmetries for functions on the unit interval $\mathcal{I} = [0, 1]$. We then show that up to this reflection symmetry, we get recursive algorithms which in turn produce infinite families of orthonormal bases localized in $L^2(0, 1)$. Our scheme is as versatile as the more traditional spline constructions; see, for example, [Wic94]. But while singly generated spline systems do not produce orthonormal bases, each of our present algorithms does. Moreover our scheme is adapted to the fixed unit interval $\mathcal{I} = [0, 1]$ while the more traditional wavelet-based wavepackets are designed for the construction of orthonormal bases in $L^2(\mathbb{R})$; see [CoWi92]. And if the starting functions are of compact support, the size of the support reaches outside the unit interval $[0, 1]$.

The more traditional approaches to basis algorithms further have limited the libraries of functions to be used at the initial step of the recursion. We get around this here by identifying a set of symmetry conditions that may be imposed on two functions ψ_0 and ψ_1 in the Hilbert space $L^2(0, 1)$; see Fig. 1 for an illustration in the simplest case. Our algorithm is then based on a certain matrix scaling and subdivision applied to these two functions. Hence our starting point is different from the more traditional one which begins with a scaling identity, masking coefficients, and a so-called father function φ_0 which solves the corresponding scaling identity; see [Dau92].

Our justification for the term “wavelet” in connection with the present basis is threefold:

- (a) Our functions are localized in a sense which will be made clear.
- (b) Our construction is recursive.
- (c) Our algorithm for constructing orthonormal bases starts with a prescribed and carefully selected finite system S of functions. Two operations are applied recursively to S , scaling and reflection. But note that as the algorithm runs, the reflections are scaled as well.

Definitions 1.1. A family of functions (typically an orthonormal basis) $\{\psi_n : n \geq 1\}$ in $L^2(0, 1)$ is said to be local if for all $\epsilon > 0$ there exists an $N(\epsilon) \geq 1$ so that the closed linear space spanned by $\{\psi_n : 1 \leq n \leq N(\epsilon)\}$ admits a finite subfamily of orthogonal functions $\{\varphi_n : 1 \leq n \leq M(\epsilon)\}$ total for the subspace spanned by $\{\psi_n : 1 \leq n \leq N(\epsilon)\}$ with the restriction that the Lebesgue measure of the support of each φ_n is less than ϵ .

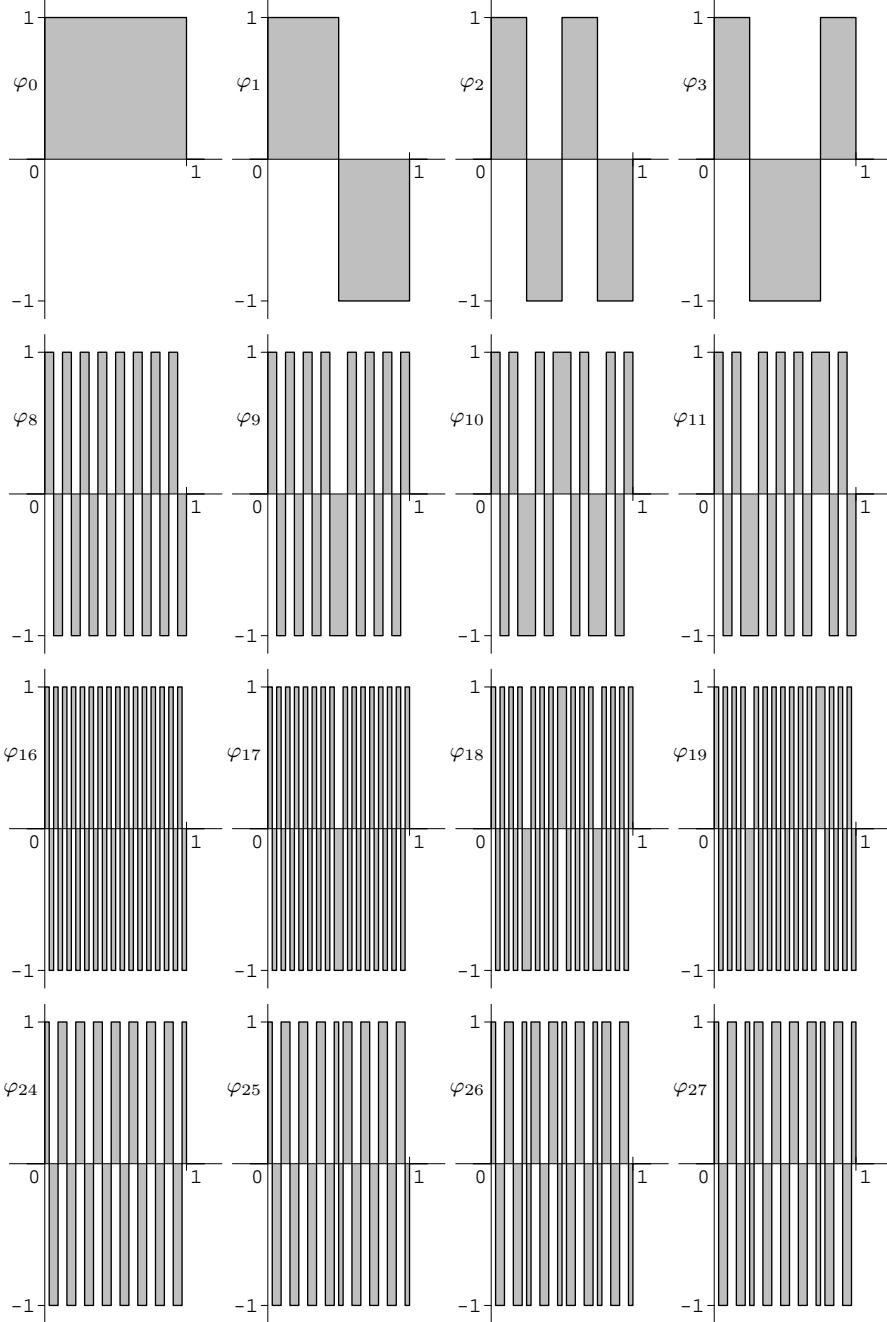


Fig. 1. The first thirty-two functions in the sequence φ_n

To understand the two discrete operations which underlie our construction, it is helpful to review a fundamental feature of wavelets in the non-standard setting of iterated function systems (IFS); see also [JoPe94, JoPe96] and Section 4 below for additional details. In the simplest of settings, i.e., that of the unit interval, the two discrete operations going into our algorithms are that of iterated scaling by 2, i.e., the system $x \rightarrow 2^m x$, modulo 1, for $m = 0, 1, 2, \dots$, and mid-point reflections. Both operations have analogues for more general IFSs, and we discuss these generalizations below.

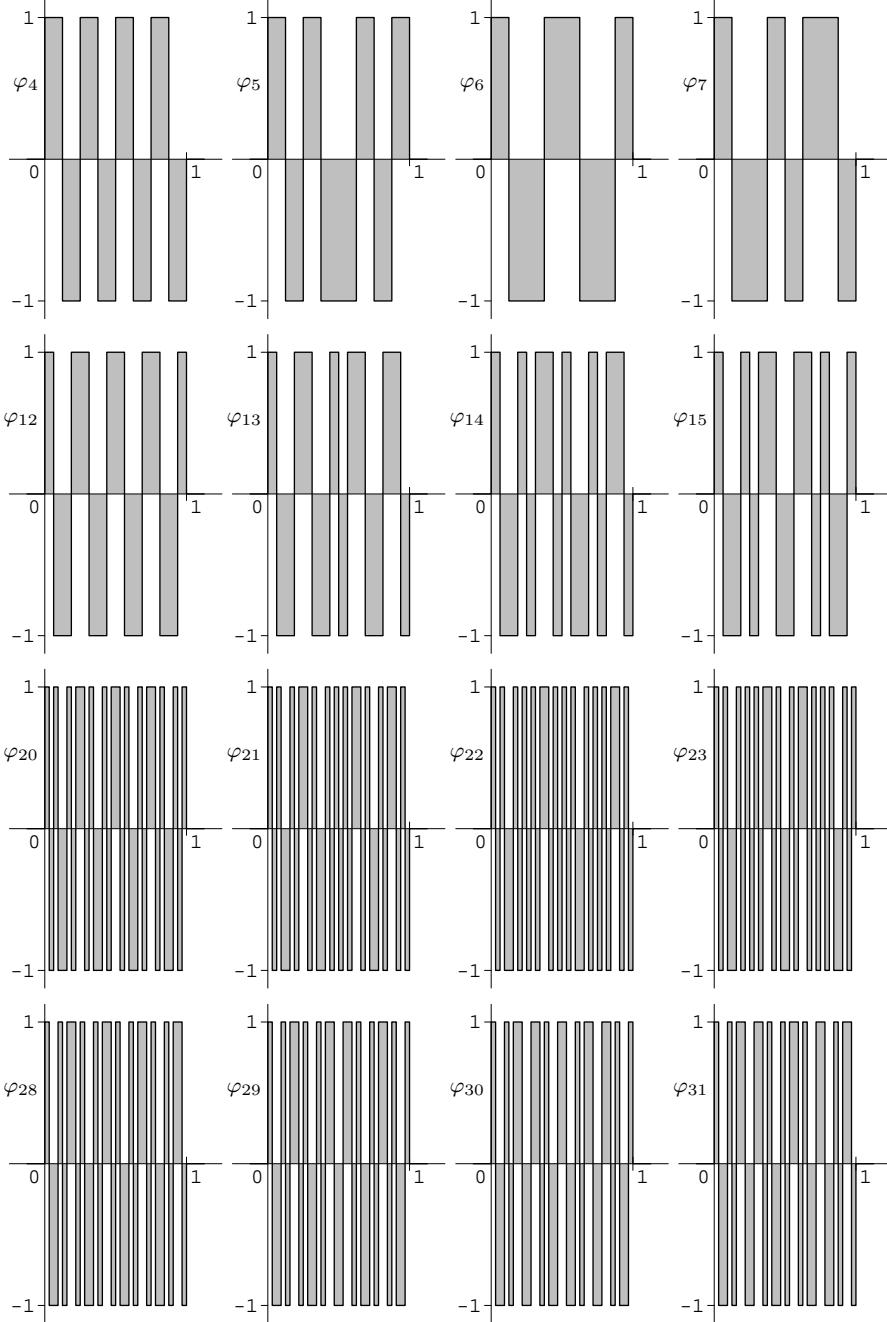


Fig. 1 (cont'd).

Our use of the terms “reflection” and “reflection symmetry” is related to, but different from, the one studied in, for example, [JoOl98]. The main difference is that in our present context, the Hilbert-space inner product is preserved after the reflection, while in [JoOl98] it is changed, i.e., it is subjected to a certain renormalization.

2 Iterated function systems

To understand our results it is useful to consider a slightly more general setup: Let d be a fixed dimension, and consider a given finite set S of affine and contractive mappings $\tau_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$. There are several interesting limits in the literature, arising from iteration of such a system $S = (\tau_i)$. The accepted terminology is “affine Iterated Function System (IFS)”. See [Hut81], [Jor06], and formulas (2.1)-(2.2) below.

Following Cantor’s middle-third construction, note that one limiting object derived from S results from recursive iterations of the individual maps in S ; it is an attractor which takes the form of a compact subset $X (= X(S))$ of \mathbb{R}^d . The set $X(S)$ often has fractal like properties: And in one dimension ($d = 1$) the deleted-middle-third Cantor set is an example, but the unit interval is one too. The other iteration limits in this context take place in the family of probability measures on \mathbb{R}^d . However when S is given, the limit measure in question (called equilibrium measure) depends further on a chosen assignment of probability weights: It turns out that, given S , and given a fixed assignment of probabilities (p_i) to the τ_i s, there is a unique equilibrium measure $\mu = \mu_{S,p}$. If $p_i > 0$ for all i , then the support of the measure $\mu_{S,p}$ is $X(S)$. Even in 1D when $X(S)$ may be the unit interval, there is a variety of measures arising from the second limit construction other than the restricted Lebesgue measure.

In this paper, we are interested in localized orthonormal bases (ONBs) in $L^2(\mu_{S,p})$. It turns out that our main issues may be best presented in the case of $d = 1$, and in the special case where the weights are uniform, i.e., $p_i = 1/N$ where N is the number of maps τ_i from the initial system S . In fact, as noted in [DuJo06b], the Hilbert space $L^2(\mu_{S,p})$ does not have ONBs of complex exponential Fourier bases in the case of non-uniform weights.

While the gist of our paper is for the unit interval, we wish to add that our construction in fact works in a more general context, that of iterated function systems (IFS) [Hut81, JoPe94]. However, it is easier to get an overview of the totality of admissible bases in the special case of the unit interval $\mathcal{I} = [0, 1]$.

We consider decomposition theory for the Hilbert space of square-integrable functions on the unit interval, both with respect to Lebesgue measure, and also with respect to a wider class of self-similar measures μ [JoPe96]. That is, we consider recursive and orthogonal decompositions for the Hilbert space $L^2(\mu)$ where μ is some self-similar measure on $[0, 1]$.

We now turn to the technical details needed in our discussion of IFSs in Sections 3 and 4 below. For the axiomatics of IFSs, see, e.g., [Hut81] and [Jor05]. A finite iterated function system is determined by a finite system of contractive endomorphisms $\tau = (\tau_i)$ in a compact metric space X . We shall

consider here a finite system consisting of N endomorphisms of a compact space X . As we will see, one of the endomorphisms will be singled out, and it will be convenient to index it by zero, i.e., the first map is τ_0 . When such a system τ is given, it is then known that there is a unique Borel probability measure μ on X such that

$$\mu = \frac{1}{N} \sum_i \mu \circ \tau_i^{-1}, \quad (2.1)$$

or equivalently,

$$\int_X f(x) d\mu(x) = \frac{1}{N} \sum_i \int f(\tau_i(x)) d\mu(x) \quad \text{for all } f \in C(X).$$

The measure μ is called the Hutchinson measure, and it is also called the balanced invariant measure of τ . In the case when X is $[0, 1]$, the unit interval, then the two maps τ_0 and τ_1 may be taken to be $x \rightarrow x/2$ and $x \rightarrow (x + 1)/2$, respectively, and μ is then the standard normalized Lebesgue measure on $[0, 1]$. For general IFSs (X, τ) , the starting point for our ONB construction in the Hilbert space $L^2(X, \mu)$ is then a specified finite set of functions (ψ_i) which satisfy a certain reflection symmetry which we proceed to describe in detail in Sections 3 and 4 below.

Let (X, d) be a compact metric space. Let $\sigma: X \rightarrow X$ be an endomorphism such that the number of elements in $\sigma^{-1}\{x\}$ is equal to N for all $x \in X$, where N , $2 \leq N < \infty$, is fixed. Iteration of branches

$$\sigma^{-1}(\{x\}) = \{y \in X \mid \sigma(y) = x\}$$

then gives rise to a combinatorial tree. If

$$\omega = (\omega_1, \omega_2, \dots) \in \Omega = \{0, 1, \dots, N - 1\}^{\mathbb{N}},$$

an associated path may be thought of as an infinite extension of finite walks

$$\tau_{\omega_n} \tau_{\omega_{n-1}} \cdots \tau_{\omega_2} \tau_{\omega_1} x$$

with starting point x , where (τ_i) , $i = 0, 1, \dots, N - 1$, is a system of Borel measurable inverses of σ , i.e.,

$$\sigma \circ \tau_i = \text{id}_X, \quad 0 \leq i \leq N - 1.$$

We assume that each (τ_i) is contractive, i.e., there exists a constant $0 < c < 1$ such that

$$d(\tau_i x, \tau_i y) \leq c d(x, y), \quad x, y \in X, \quad i \in \{0, 1, \dots, N - 1\}.$$

This contractive condition ensures that there exists a Borel probability measure μ on X such that

$$\mu = \frac{1}{N} \sum_{i=0}^{N-1} \mu \circ \tau_i^{-1}.$$

Before we state our first simple lemma we also set

$$\alpha_i = \sigma \text{ on } \tau_i(X).$$

For each $j \in \{0, 1, \dots, N-1\}$ we define a linear operator (S_j) by

$$S_j f = \sum_{k=0}^{N-1} e^{i2\pi \frac{jk}{N}} \chi_{\tau_k(X)} f \circ \alpha_k \quad (2.2)$$

for all $f \in \mathcal{H} = L^2(X, \mu)$.

Lemma 2.1. *If the operators S_j in $L^2(X, \mu)$ are given as in (2.2), then the formula for the corresponding adjoint operators S_j^* is as follows:*

$$S_j^* f = \frac{1}{N} \sum_{k=0}^{N-1} e^{-i2\pi \frac{jk}{N}} f \circ \tau_k \quad (2.3)$$

PROOF. We leave the details to the reader. They are based on the τ_i -equipartition property for the measure μ . See (2.1), and also [JoPe94]. \square

In the next lemma we show that the operators S_j from (2.2) define a representation of the Cuntz C^* -algebra. As noted in [JoPe94, JoPe96], this C^* -algebra has found a variety of uses in iterated function systems (IFS), and in approximation theory. Its use in analysis and in physics was initiated in the paper [Cu77]. For each IFS with N endomorphisms, there is an associated representation of the Cuntz algebra \mathcal{O}_N with N generators S_j . These generators are also called the fundamental isometries. (When we say “isometry”, we are here referring to the Hilbert space $L^2(X, \mu)$.) From [Cu77] we further know that specifying a representation of \mathcal{O}_N is equivalent to specifying a system of N fundamental isometries.

Lemma 2.2. *The system of operators (S_j) from (2.2) defines a representation of the Cuntz algebra \mathcal{O}_N . We write $(S_j) \in \text{Rep}(\mathcal{O}_N, \mathcal{H})$, i.e.,*

$$S_j^* S_k = \delta_{j,k} \mathbb{1}_{\mathcal{H}}, \quad \sum_{k=0}^{N-1} S_k S_k^* = \mathbb{1}_{\mathcal{H}}. \quad (2.4)$$

PROOF. Again, we leave the details to the reader. The argument uses the previous lemma combined with the axioms for the system (τ_j) and the associated measure μ , outlined above.

Returning to formula (2.2) and setting

$$m_j(x) = \sum_{k=0}^{N-1} e^{i2\pi \frac{jk}{N}} \chi_{\tau_k(X)}(x),$$

we see that (2.2) may be rewritten in the form

$$(S_j f)(x) = m_j(x) f(\sigma(x)) \quad \text{for } f \in L^2(X, \mu), x \in X. \quad (2.5)$$

Hence using an idea from [BJMP05], we then note that the Cuntz relations (2.4) for the operators (S_j) in (2.5) are equivalent to the assertion that the $N \times N$ matrix function U given by

$$U(x) = (U_{j,k}(x))_{j,k=0}^{N-1} := \frac{1}{\sqrt{N}} \left(m_j(\tau_k(x)) \right)_{j,k=0}^{N-1}$$

takes values in the group $U_N(\mathbb{C})$ of all $N \times N$ unitary matrices.

We now check that $UU^* = \mathbb{1}_N$. Substituting $m_j(\tau_k(x)) = e^{i2\pi \frac{jk}{N}}$ into the formula for the matrix product, we get

$$\sum_{k=0}^{N-1} U_{j,k}(x) \overline{U_{l,k}(x)} = \frac{1}{N} \sum_{k=0}^{N-1} e^{i2\pi \frac{(j-l)k}{N}} = \delta_{j,l}. \quad \square$$

Remark 2.3. There is a variety of representations of the Cuntz algebra \mathcal{O}_N , other than those that come naturally from IFSs (as in (2.2) and the lemma). In fact by a theorem of Glimm [Gli60], the set of equivalence classes of irreducible representations of \mathcal{O}_N cannot be parametrized by a Borel cross section.

Our focus in this paper is the search for orthonormal bases (ONBs) in the Hilbert space $L^2(\mu)$ associated naturally with a fixed contractive IFS. Once an ONB is chosen, it may be used in the analogue-to-digital (A-to-D) conversion of signals. Conversely, for a given signal (read, $f \in L^2(\mu)$) there is a variety of ONBs, and a choice must be made; for example, we may wish to minimize the entropy (information loss),

$$\varepsilon^{(\psi)}(f) = - \sum_k |\langle \psi_k | f \rangle|^2 \ln |\langle \psi_k | f \rangle|^2,$$

where (ψ_k) is the chosen ONB. (Here, we shall normalize f , i.e., assume $\|f\|_{L^2(\mu)} = 1$.)

We make use of representations in two ways: (1) as a first step toward the determination of an ONB; and (2) for entropy computations, even when an ONB is not known.

The details on (1) will be presented in the next section; and we now briefly discuss (2).

As illustrated in the lemma below, our use of the fundamental isometries (S_i) in a particular representation of \mathcal{O}_N in some Hilbert space \mathcal{H} immediately yields scales of mutually orthogonal subspaces of \mathcal{H} , or equivalently, mutually orthogonal projections which constitute partition of the identity operator $\mathbb{1}$ in \mathcal{H} . Example: the projections $P_i := S_i S_i^*$ are orthogonal and satisfy

$$\sum P_i = \mathbb{1}.$$

As stressed in for example [Kri05] and [JKK05], the use of subspaces as opposed to vectors is significant for the design of quantum algorithms such as quantum error-correction codes. The reason is that direct operations on individual vectors in the Hilbert space \mathcal{H} typically would destroy the quantum states on which the algorithm is operating. The quantum states (or qubits) might represent polarized photons. (Recall that in the conventions of quantum theory, unit-vectors in the underlying Hilbert space \mathcal{H} represent quantum states!) Our use of representations of \mathcal{O}_N lets the algorithms act on subspaces in \mathcal{H} , and as a result leave the individual quantum states intact. The intrinsic orthogonality of the subspaces is what yields quantum channels; see [JKK05].

Lemma 2.4. *Let $(S_i) \in \text{Rep}(\mathcal{O}_N, \mathcal{H})$ be a representation of \mathcal{O}_N acting on a Hilbert space \mathcal{H} . Let \mathcal{M}_k denote the set of all multi-indices $J = (j_1 j_2 \dots j_k)$, where each j_i is in $\{0, 1, \dots, N-1\}$. Then for each k ,*

$$S_J := S_{j_1} \cdots S_{j_k}, \quad J \in \mathcal{M}_k,$$

is a representation of \mathcal{O}_{N^k} ; in particular, $P_J := S_J S_J^$ yields a commuting family of mutually orthogonal projections in \mathcal{H} , i.e.,*

$$P_J P_{J'} = \delta_{J,J'} P_J, \quad J \in \mathcal{M}_k,$$

and

$$\sum_{J \in \mathcal{M}_k} P_J = \mathbb{1}_{\mathcal{H}}.$$

For f in \mathcal{H} , $\|f\| = 1$, the entropy number

$$\varepsilon_k(f) := - \sum_{J \in \mathcal{M}_k} \|P_J f\|^2 \ln \|P_J f\|^2$$

satisfies

$$\varepsilon_{k+1}(f) = \varepsilon_1(f) + \sum_i \|S_i^* f\|^2 \varepsilon_k \left(\frac{P_i f}{\|P_i f\|} \right),$$

where $P_i = S_i S_i^$ and*

$$\varepsilon_1(f) := - \sum_{i=0}^{N-1} \|S_i^* f\|^2 \ln \|S_i^* f\|^2.$$

PROOF. The steps in the proof of the lemma are straightforward, and we leave them to the reader. \square

Example 2.5. Let $X = [0, 1]$ and let $\sigma: X \rightarrow X$ be the endomorphism $\sigma(x) = 2x \bmod 1$. Here $\tau_0(x) = \frac{x}{2}$ and $\tau_1(x) = \frac{x+1}{2}$ are two maps for which $\sigma \circ \tau_i(x) = x$, $i = 0, 1$. Let $\alpha_0(x) = 2x\chi_{[0, \frac{1}{2}]}$ and $\alpha_1(x) = (2x-1)\chi_{[\frac{1}{2}, 1]}$; then

$$\begin{aligned} S_0 f &= f \circ \alpha_0 + f \circ \alpha_1, & S_0^* f &= \frac{1}{2}(f \circ \tau_0 + f \circ \tau_1), \\ S_1 f &= f \circ \alpha_0 - f \circ \alpha_1, & S_1^* f &= \frac{1}{2}(f \circ \tau_0 - f \circ \tau_1). \end{aligned}$$

We choose unit vectors φ, ψ so that $S_1^* \varphi = 0$ and $S_0^* \psi = 0$. By Cuntz's relation (2.4) we get $S_0 S_0^* \varphi = \varphi$ and $S_1 S_1^* \psi = \psi$. Thus we get $\langle \varphi | \psi \rangle = 0$. By our construction,

$$S_0^* \psi = 0$$

if

$$\psi \circ \tau_0 = -\psi \circ \tau_1,$$

i.e.,

$$\psi\left(\frac{x}{2}\right) = -\psi\left(\frac{x+1}{2}\right),$$

or equivalently,

$$\psi(x) = -\psi\left(\frac{2x+1}{2}\right) = -\psi\left(x + \frac{1}{2}\right) \quad \text{for all } 0 \leq x \leq \frac{1}{2}.$$

Lemma 2.6. *Let f be an element of $L^2(0, 1)$. Then the following statements are equivalent:*

- (a) $S_0^* f = 0$;
- (b) $f(x) = -f\left(\frac{1}{2} + x\right)$ for all $0 \leq x \leq \frac{1}{2}$;
- (c) $f(x) = \sum_{n \geq 0} a_n \cos(2\pi(2n+1)x) + b_n \sin(2\pi(2n+1)x)$.

Moreover $S_1 S_1^*$ is the projection onto the closed subspace

$$\left\{ f \in L^2(0, 1) \mid f(x) = -f\left(\frac{1}{2} + x\right), 0 \leq x \leq \frac{1}{2} \right\}.$$

Similarly the following statements are equivalent:

- (a) $S_1^* f = 0$;
- (b) $f(x) = f\left(\frac{1}{2} + x\right)$ for all $0 \leq x \leq \frac{1}{2}$;
- (c) $f(x) = \sum_{n \geq 0} a_n \cos(2\pi(2n)x) + b_n \sin(2\pi(2n)x)$.

Moreover $S_0S_0^*$ is the projection onto the closed subspace

$$\left\{ f \in L^2(0, 1) \mid f(x) = f\left(\frac{1}{2} + x\right), 0 \leq x \leq \frac{1}{2} \right\}.$$

PROOF. Equivalence of statements (a) and (b) is obvious. That (c) implies (b) is routine as

$$\cos(x + (2m + 1)\pi) = -\cos(x)$$

and

$$\sin(x + (2m + 1)\pi) = -\sin(x)$$

for any $m \geq 0$. We will prove now that (a) implies (c). To that end, for any $n \geq 0$ we set

$$c_n(f) = \int_0^1 \cos(2\pi nx) f(x) dx$$

and

$$s_n(f) = \int_0^1 \sin(2\pi nx) f(x) dx.$$

A simple computation shows that $c_n(S_0^* f) = c_{2n}(f)$ and $s_n(S_0^* f) = s_{2n}(f)$ for all $n \geq 0$. Thus $c_{2n}(f) = s_{2n}(f) = 0$ if $S_0^* f = 0$. The last statement follows since by the Cuntz relations, $S_0^* f = 0$ if and only if $S_1 S_1^* f = f$.

For equivalence of statements in the second set, we note that by the Cuntz relations, $\langle f \mid g \rangle = 0$ whenever $S_0^* f = 0$ and $S_1^* g = 0$. Thus, the equivalence of the second set of statements follows from that of the first set of statements. \square

Notation 2.7. We shall need the subspaces

$$\mathcal{K}_0 := \{ \psi \mid S_0^* \psi = 0 \}$$

and

$$\{ S_I \psi \mid \Sigma(I) \leq 1 \}.$$

The symbol $\Sigma(I)$ in the second subspace $\{ S_I \psi \mid \Sigma(I) \leq 1 \}$, $\Sigma(I) = \sum_k i_k$, is defined as all multi-indices I such that $\Sigma(I) \leq 1$. It is motivated as follows: In building ONBs, the aim is to start with a conveniently chosen function ψ , and then to construct the rest from recursively applying monomials in the generators S_i (chosen from a particular \mathcal{O}_N representation). This notation allows us to keep track of the combined system of relations in step-size of length one.

For any ψ with $S_0^* \psi = 0$, we have

$$\begin{aligned} \int_{\frac{1}{2}}^1 \psi(x) \psi(2x - 1) dx &= \int_0^{\frac{1}{2}} \psi\left(\frac{1}{2} + x\right) \psi(2x) dx \\ &= - \int_0^{\frac{1}{2}} \psi(x) \psi(2x) dx. \end{aligned}$$

Thus we have

$$\begin{aligned}\langle \psi | S_1\psi \rangle &= \int_0^{\frac{1}{2}} \psi(x)\psi(2x) dx - \int_{\frac{1}{2}}^1 \psi(x)\psi(2x-1) dx \\ &= 2 \int_0^{\frac{1}{2}} \psi(x)\psi(2x) dx.\end{aligned}$$

More generally for any two elements $\psi^{(1)}, \psi^{(2)}$ of the subspace

$$\mathcal{K}_0 := \{ \psi \mid S_0^* \psi = 0 \},$$

we have

$$\langle \psi^{(1)} | S_1\psi^{(2)} \rangle = 2 \int_0^{\frac{1}{2}} \psi^{(1)}(x)\psi^{(2)}(2x) dx.$$

Any vector $\psi \in \mathcal{H}$ such that $S_0^*\psi = 0$ and $\langle \psi | S_1 S_0^m \psi \rangle = 0$ for all $m \geq 0$ is called a *generating vector* for the closed linear span of the vectors $\{ S_I \psi \mid \Sigma(I) \leq 1 \}$ (see Notation 2.7 above and Lemma 2.10 below for the notation used here). A subspace \mathcal{K} of \mathcal{K}_0 is called a *basis space* for generating vectors if it is a maximal family of vectors that satisfies the following mutual relation:

$$\langle \psi | S_1 S_0^m \psi' \rangle = 0, \quad m \geq 0,$$

for all $\psi, \psi' \in \mathcal{K}$. Existence of a such a maximal family of vectors follows by Zorn's lemma. It is simple to note that \mathcal{K} is a subspace of \mathcal{K}_0 .

Definition 2.8. A maximal subspace \mathcal{K} as above will be called a *basis space*.

We now turn to a concrete representation of the subspaces \mathcal{K}_0 and \mathcal{K} in $L^2(0, 1)$ which were outlined above.

Our identification of subspaces \mathcal{K}_0 and associated orthonormal bases is by a certain algorithmic procedure. Below we illustrate our recursive construction in one particular example: In our construction we begin with the representation of \mathcal{O}_2 from Lemma 2.2, and we give a natural and orthogonal subspace decomposition of the Hilbert space $L^2(0, 1)$. Using this, we then show how an associated recursive basis may be realized. In our example, we start with a family of sine functions, normalized to have period one. We then aim for an orthonormal basis (ONB) when the Hilbert space $L^2(0, 1)$ is defined from the restriction of Lebesgue measure to the unit interval $\mathcal{I} = [0, 1]$. (Other self-similar measures will be considered later!) Our example will further serve to illustrate the reflection operations which we will encounter later in a more general context of self-similar systems.

As outlined before, the idea is to start our recursion from two prescribed functions ψ_0 and ψ_1 . Here we take ψ_0 to be the constant function “one” on $\mathcal{I} = [0, 1]$; and we choose $\psi_1(x) := s(x) := \sin(2\pi x)$. The recursion will be

as outlined above: The idea is to recursively determine the pair of functions (ψ_{2n}, ψ_{2n+1}) from sampled subdivisions of ψ_n for each $n \geq 1$. (Note that our recursion does not begin with $n = 0$.) Notation: Set $s_n(x) := \sin(2\pi nx)$, $n = 1, 2, \dots$.

Lemma 2.9. *For every odd integer n , the function s_n is in \mathcal{K}_0 . Moreover, if n is even, then $S_0^* s_n$ is non-zero in $L^2(0, 1)$.*

PROOF. We begin by setting $s(x) = \sin(2\pi x)$, and for any integer $n \geq 1$ we also set $s_n(x) = s(nx) = \sin(2\pi nx)$, $x \in [0, 1]$. As

$$\begin{aligned} s_1\left(\frac{1+x}{2}\right) &= s\left(\frac{1+x}{2}\right) = \sin(\pi(1+x)) \\ &= -\sin(\pi x) = -s_1\left(\frac{x}{2}\right), \end{aligned}$$

we have $s_1 \in \mathcal{K}_0$. For any odd integer, i.e., $n = 2m + 1$, we check that

$$\begin{aligned} s_n\left(\frac{1+x}{2}\right) &= \sin(\pi(2m+1)(1+x)) = \sin(\pi(2m+1)x + (2m+1)\pi) \\ &= -\sin(\pi(2m+1)x) = -s_n\left(\frac{x}{2}\right). \quad \square \end{aligned}$$

We shall need the following additional facts about the functions s_n . For any two integers $m, n \geq 0$, we have

$$\begin{aligned} \langle s_m | S_1 s_n \rangle &= 2 \int_0^{\frac{1}{2}} s_m(x) s_n(2x) dx \\ &= \int_0^{\frac{1}{2}} [\cos(2\pi(m-2n)x) - \cos(2\pi(m+2n)x)] dx \\ &= \left[\frac{\sin(2\pi(m-2n)x)}{2\pi(m-2n)} - \frac{\sin(2\pi(m+2n)x)}{2\pi(m+2n)} \right] \Big|_0^{\frac{1}{2}} = 0 \end{aligned}$$

for $m \neq 2n$. For $m = 2n$, we also check that the integral is 0. Since $S_0^m s_n = s_{2m} s_n$, we also get $\langle s_m | S_1 S_0^m s_n \rangle = 0$. Hence, $\{s_{2n+1}(x) \mid n \geq 0\}$ is a family of orthonormal vectors in a basis space. One natural question that we face now: Is it a maximal family, i.e., is it a basis space? We answer this in the affirmative in the remaining part of this section.

Lemma 2.10. *Let ψ be a unit vector such that*

$$S_0^* \psi = 0 \quad \text{and} \quad \langle \psi | S_1(S_0)^m \psi \rangle = 0 \quad \text{for all } m \geq 0,$$

but

$$\langle \psi | S_1(S_0)^{m'} S_1(S_0)^m \psi \rangle \neq 0 \quad \text{for all } m, m' \neq 0.$$

Then it follows that the family of vectors

$$\{ S_I \psi \mid \Sigma(I) \leq 1 \},$$

where

$$\Sigma(I) = \sum_k i_k,$$

and

$$I = (i_1 i_2 \dots i_k), \quad i_k \in \{0, 1\},$$

is a maximal family of orthonormal vectors. The closed subspace $\mathcal{H}(\psi)$ spanned by the vectors $\{ S_I \psi \mid |I| < \infty \}$ is invariant under both of the operators S_0 and S_0^* .

PROOF. Since $S_0^* \psi = 0$, it is simple to verify by the Cuntz relations that $(S_0)^m \psi$ is orthogonal to $(S_0)^n \psi$ for $m \neq n \geq 0$, where by convention $(S_0)^0 = \mathbb{1}$. As $S_1^* S_0 = 0$ and $S_0^* \psi = 0$, we also check that $(S_0)^{m'} S_1 (S_0)^m \psi$ is orthogonal to $(S_0)^{n'} S_1 (S_0)^n \psi$ and $(S_0)^{n'} \psi$ for all $m' \neq n'$ and $n, n' \geq 0$. Thus we are left to check for $m' = n'$. In such case orthogonality follows by our hypothesis that $\langle \psi \mid S_1 (S_0)^m \psi \rangle = 0$ for all $m \geq 0$. It is clear that the vector space generated by these vectors is both S_0 - and S_0^* -invariant. The maximal property is also evident. See Section 3 below for details. \square

Terminology: \mathcal{M} will denote the set of all finite multi-indices.

Proposition 2.11. *Let (S_i) be the irreducible representation of \mathcal{O}_2 as in Example 2.5, and let \mathcal{K} be a basis space. For a vector ψ use the notation $\mathcal{H}(\psi)$ as in Lemma 2.10 for the closed subspace spanned by the vectors $\{ S_I \psi \mid |I| < \infty \}$. Then the following hold:*

- (a) *For each unit vector $\psi \in \mathcal{K}$, the vectors in the family $\{ S_I \psi \mid \Sigma(I) \leq 1 \}$ are orthonormal, i.e., norm one, and mutually orthogonal.*
- (b) *For any two orthogonal unit vectors $\psi^{(1)}, \psi^{(2)} \in \mathcal{K}$, $\mathcal{H}(\psi^{(1)})$ is orthogonal to $\mathcal{H}(\psi^{(2)})$.*

PROOF. Proof is routine as in Lemma 2.10. \square

Lemma 2.12. (Haar–Walsh, Fig. 1 above, [Haa10, Wal23]) *Let $\mathcal{H} = L^2(0, 1)$ and let $\varphi_0 = \chi_{[0,1]}$. Then the recursive system*

$$\begin{aligned} \varphi_{2n}(x) &= \varphi_n(2x) + \varphi_n(2x - 1), \\ \varphi_{2n+1}(x) &= \varphi_n(2x) - \varphi_n(2x - 1) \end{aligned}$$

for $n \geq 0$ defines an orthonormal basis for \mathcal{H} .

PROOF. We set S_0, S_1 as in Example 2.5. We consider the Hardy space \mathcal{H}_+ given by

$$f(z) = \sum_{n \geq 0} c_n z^n, \quad z \in \mathbb{T}^1 = \{ z \in \mathbb{C} \mid |z| = 1 \},$$

for $(c_n) \in l^2$ and $\|f\|^2 = \sum_{n \geq 0} |c_n|^2$. We also set $e_n(z) = z^n$ and

$$\begin{aligned} \tilde{S}_0 f(z) &= f(z^2), \\ \tilde{S}_1 f(z) &= z f(z^2) \end{aligned}$$

for all $f \in \mathcal{H}_+$ and $z \in \mathbb{T}^1$. We have $S_0^* \varphi_0 = \varphi_0$ and $\tilde{S}_0^* e_0 = e_0$. Now, defining $W: \mathcal{H}_+ \rightarrow \mathcal{H}$ by $W e_n = \varphi_n$, we verify that

$$W \tilde{S}_i = S_i W \quad \text{for } i \in \{0, 1\}.$$

If $n = j_1 + j_2 2 + \cdots + j_k 2^k$ where $j_r \in \{0, 1\}$ is the dyadic representation of an integer $n \geq 0$, it follows that $W e_n = W \tilde{S}_{j_1} \tilde{S}_{j_2} \cdots \tilde{S}_{j_k} e_0 = S_{j_1} S_{j_2} \cdots S_{j_k} \varphi_0 = \varphi_n$, and the result follows. In the last step we are using the fact (details in [Jor05]) that the two \mathcal{O}_2 representations are irreducible; so the intertwining operator W is a constant times a unitary. \square

Definitions 2.13. (a) For $\psi \in \mathcal{H} \setminus \{0\}$, set

$$\begin{aligned} K(\psi) := \min \{ K \in \mathcal{M} \mid \text{s.t. } \langle S_{J_1} \psi \mid S_{J_2} \psi \rangle = 0 \ \forall J_i \not\leq K, \ J_1 \neq J_2, \\ \text{and } \exists J \not\leq K \text{ s.t. } \langle S_J \psi \mid S_K \psi \rangle \neq 0 \}. \end{aligned}$$

(b) Further, define

$$\mathcal{H}(\psi) := \text{closed span} \{ S_0^m S_J \psi \mid m \in \mathbb{N}_0, \ J \not\leq K(\psi) \}. \quad (2.6)$$

(c) A family of vectors (ψ_n) , $\|\psi_n\| = 1$, is said to be *maximal* and orthogonal if (i) the corresponding subspaces $\mathcal{H}(\psi_n)$ are mutually orthogonal, (ii) $\mathcal{H}(\psi_n) \perp \varphi$, and (iii) they are not part of a bigger such family.

Theorem 2.14. *Let \mathcal{K} be a basis space, and let $\{ \psi_n \mid n \geq 1 \}$ be an orthonormal basis for \mathcal{K} . Further, let \mathcal{H}_n be a maximal family of subspaces associated with (ψ_n) in the sense of Definition 2.13(c). Then*

$$\mathcal{H} = \mathbb{C} \varphi \bigoplus_{n \geq 1} \mathcal{H}_n.$$

Before turning to the proof, we will need some preliminaries which are included in the remarks below. The proof will then be resumed at the end of Section 3 below.

The reasoning in the proof of the theorem has the following two parts in rough outline: Firstly, the argument for orthogonality of the spaces and the vectors which go into our basis construction is largely combinatorial, and it is sketched in Remarks 2.15 below.

Secondly, we must prove that the functions which are produced by the algorithm form a total family in $L^2(0, 1)$, i.e., that these vectors span a dense subspace in $L^2(0, 1)$. Recall our algorithm starts with the two functions $\psi_0 = \varphi =$ the constant function “one”, and $\psi_1(x) = s_1(x) = \sin(2\pi x)$. Using the sine functions from Lemma 2.9, and our representation of \mathcal{O}_2 from Lemmas 2.2 and 2.10, we then organize a system of orthogonal functions in each of the closed subspaces \mathcal{H}_n from the conclusion of the theorem. Our assertion is that this orthonormal family of vectors is total. Our argument going into the proof of this is structured as follows (details in Section 3): Suppose some f in $L^2(0, 1)$ is in the orthogonal complement of the family. We then show that both vectors f and S_1^*f must have Fourier expansions consisting only of cosine functions. Because of the reflection built into the operator S_1^* , we conclude that this is only possible if f is zero.

Remarks 2.15. Our algorithm may be applied both to existing wavelets, and to new ones as well. Consider for example the Haar–Walsh sequence

$$\{ \varphi_n \mid n \geq 0 \} \subseteq L^2(0, 1)$$

defined as in Lemma 2.12. For any multi-indices $J = (j_1 j_2 \dots j_n) \in \mathcal{M}$, $j_i \in \{0, 1\}$, consider the orthogonal family $S_J \varphi$ and set

$$\mathcal{M}_{ev} = \{ J \in \mathcal{M} \mid \Sigma(J) = 0, 2, 4, 6, \dots \}$$

and

$$\mathcal{M}_1 = \{ J \in \mathcal{M} \mid \Sigma(J) \leq 1 \}. \quad (2.7)$$

Here a number N is fixed and our multi-indices are built from the alphabet $\{0, 1, \dots, N - 1\}$. Further we define $\Sigma(J)$ to be $\sum_1^n j_i$.

For any $J \in \mathcal{M}_{ev}$, we set

$$\mathcal{H}(S_J \psi) = \text{closed span}\{ S_{KJ} \psi \mid K \in \mathcal{M}_1 \}.$$

We claim that

$$\sum_{J \in \mathcal{M}_{ev}}^{\oplus} \mathcal{H}(S_J \psi) \oplus \mathbb{C} \varphi_0 = L^2(0, 1) \quad (2.8)$$

and the vectors appearing in (2.8) are orthonormal. Note that the first few terms in the system of closed subspaces in (2.8) are

$$\mathcal{H}(\varphi_1) \oplus \mathcal{H}(\varphi_7) \oplus \mathcal{H}(\varphi_{11}) \oplus \mathcal{H}(\varphi_{13}) \oplus \dots.$$

Specifically,

$$\begin{aligned}\varphi_7 &= S_1^2\psi, \\ \varphi_{11} &= S_1^2S_0\psi, \\ \varphi_{13} &= S_1S_0S_1\psi, \\ &\vdots\end{aligned}$$

We now turn to our basis constructions.

In brief outline (Proof of Theorem 2.14 Part 1): The starting point is a fixed IFS with N endomorphisms, and an associated representation (S_i) of \mathcal{O}_N in $L^2(\mu)$. Although it would seem natural to begin with cyclic subspaces in $L^2(\mu)$, care must be exercised in order to guarantee orthogonality. Our starting point will be two orthogonal (and carefully selected) vectors φ and ψ satisfying $S_0\varphi = \varphi$, and $S_0^*\psi = 0$. (Note that orthogonality of φ and ψ is implied by these relations.) Then using the representation, new vectors φ are constructed to make up part of an ONB. The new vectors result from applying operator monomials in the generators S_i to ψ . So operator monomials are applied to the single vector ψ , and the monomials are selected with view to orthogonality. The various new orthogonal vectors φ are assigned subscripts according to the particular operator monomial which is applied to ψ . The process is then repeated inductively on additional vectors ψ as needed for creating a full ONB.

3 Irreducible representations of \mathcal{O}_N

Let $N \in \mathbb{N}$, $N \geq 2$, and let \mathcal{H} be a separable Hilbert space. Let $\{S_i \mid 0 \leq i < N\}$ be a system of isometries in \mathcal{H} which define an irreducible representation of the Cuntz algebra \mathcal{O}_N . We shall further assume that there is some $\varphi \in \mathcal{H}$ such that $\|\varphi\| = 1$ and $S_0\varphi = \varphi$. (Note that then $S_0^*\varphi = \varphi$ as well.)

Set $\mathcal{M} = \mathcal{M}_N$ (= the set of all finite multi-indices $J = (j_0 j_1 \dots j_p)$ where $j_i \in \{0, 1, \dots, N-1\}$). Since

$$(j_0 j_1 \dots j_p) \longmapsto j_0 + j_1 N + \dots + j_p N^p$$

defines a bijection of \mathcal{M}_N onto $\mathbb{N}_0 = \{0, 1, \dots\}$, \mathcal{M}_N acquires an order induced from the natural order on \mathbb{N}_0 .

Theorem 3.1. *Let N be an integer ≥ 2 , let \mathcal{H} be a Hilbert space, and let $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, be chosen such that $S_0\varphi = \varphi$, where $(S_i)_{i=0}^{N-1}$ is a given irreducible representation of \mathcal{O}_N on \mathcal{H} . Then there is a maximal family $(\psi_n)_{n \geq 1}$, such that*

$\|\psi_n\| = 1$, $S_0^* \psi_n = 0$; and

$$\mathcal{H} = \mathbb{C} \varphi \oplus \sum_{n \geq 1}^{\oplus} \mathcal{H}(\psi_n). \quad (3.1)$$

Note, every maximal family will satisfy (3.1).

Before starting the proof, we need a lemma.

Lemma 3.2. *Let $\psi \in \mathcal{H} \setminus \{0\}$. Assumptions as above, including $S_0^* \psi = 0$. Then the corresponding subspace $\mathcal{H}(\psi)$ is invariant under both of the operators S_0 and S_0^* .*

PROOF. By (2.6), $\mathcal{H}(\psi)$ is the closed span of $\{S_0^m S_J \psi \mid m \in \mathbb{N}_0, J \leq K(\psi)\}$. Since $S_0^* \psi = 0$, the only non-trivial case to consider is the set of vectors $S_0^* S_J \psi$ when $J \neq \emptyset$ and $J \leq K(\psi)$. But note that $S_0^* S_J \psi = S_{J'} \psi$ where $J' \leq K(\psi)$. This concludes the proof. \square

PROOF OF THEOREM 3.1 Using Zorn's lemma, we may select a maximal family (ψ_n) as stated in the theorem. When it is chosen, our claim is that then (3.1) holds.

We finish the argument by assuming that the orthocomplement \mathcal{L} of $\mathbb{C} \varphi \oplus \sum_{n \geq 1} \mathcal{H}(\psi_n)$, $\mathcal{L} = \{\mathbb{C} \varphi \oplus \sum_{n \geq 1} \mathcal{H}(\psi_n)\}^{\perp} = \mathcal{H} \ominus \{\mathbb{C} \varphi \oplus \sum_{n \geq 1} \mathcal{H}(\psi_n)\}$, is nonzero, and then derive a contradiction.

By Lemma 2.2, there are two cases for the projection $S_0 S_0^*$:

Case 1. $S_0 S_0^*|_{\mathcal{L}} \leq \mathbb{1}_{\mathcal{L}}$; and

Case 2. $S_0 S_0^*|_{\mathcal{L}} = \mathbb{1}_{\mathcal{L}}$.

Assuming Case 1, there is some $\psi \in \mathcal{L} \setminus \{0\}$ such that $S_0^* \psi = 0$. Following the argument from the proof of Theorem 2.14, we now prove that

$$\mathcal{H}(\psi) \perp \mathcal{H}(\psi_n) \quad \text{for all } n \geq 1;$$

and this then contradicts maximality of the family (ψ_n) .

Assuming Case 2, and $\mathcal{L} \neq 0$, we get

$$S_i S_i^* \psi = 0 \quad \text{for all } i, 1 \leq i < N-1, \text{ and all } \psi \in \mathcal{L}.$$

This follows from

$$\sum_{i=0}^{N-1} S_i S_i^* \psi = \psi.$$

We conclude that $S_i^* \psi = 0$ for $i = 1, 2, \dots, N - 1$.

Now use the properties of the vector φ to conclude that

$$\langle S_J^* \psi \mid \varphi \rangle = \langle \psi \mid S_J \varphi \rangle = 0 \quad \text{for all } J \in \mathcal{M}. \quad (3.2)$$

We have $S_0 \varphi = S_0^* \varphi = \varphi$, and therefore $S_i^* \varphi = 0$ for $i \geq 1$. As a result, the closed subspace spanned by $\{S_J \varphi \mid J \in \mathcal{M}_N\}$ is invariant under \mathcal{O}_N . Since $\varphi \neq 0$, this subspace must be all of \mathcal{H} , and we get $\psi = 0$ from (3.2), which is a contradiction.

This concludes the proof of the theorem. \square

Proof of Theorem 2.14 concluded. We have chosen to introduce the tools of Cuntz algebra representations before the conclusion of the proof of Theorem 2.14. Once this is available, the reader will easily be able to establish the existence of a maximal family of subspaces as in the conclusion of Theorem 2.14, following the reasoning above (in the proof of Theorem 3.1) using Zorn, and in the same way dividing the analysis up into two cases.

4 A scale-4 Cantor set

In this section we continue work on harmonic analysis for iterated-function-system fractals started in [JoPe98] and continued in [DuJo06b]. Our present basis constructions differ from the earlier ones mainly in their emphasis on the use of “localized” functions.

We begin the section with an outline of a new application of our general algorithm from Theorem 3.1 to the case of the scale-4 Cantor set $X = X_4$ on the real line \mathbb{R} . This Cantor set was studied in [JoPe98]. We begin with a brief review of X_4 and its associated measure $\mu = \mu_4$: First note that by [Hut81], or by direct computation, there is a unique Borel probability measure μ on \mathbb{R} satisfying

$$\int f d\mu = \frac{1}{2} \left(\int f \left(\frac{x}{4} \right) d\mu(x) + \int f \left(\frac{x+2}{4} \right) d\mu(x) \right) \quad (4.1)$$

for all bounded continuous functions f . Its support X is the unique compact subset of \mathbb{R} satisfying $X = \tau_0(X) \cup \tau_1(X)$ where

$$\tau_0(x) = \frac{x}{4}, \quad \tau_1(x) = \frac{x+2}{4}. \quad (4.2)$$

The set X is sketched in Fig. 2. Specifically, $X = \{ \sum_{i=1}^{\infty} k_i / 4^i \mid k_i \in \{0, 2\} \}$. When fractions inside $[0, 1]$ are written in base 4, X results as a Cantor set

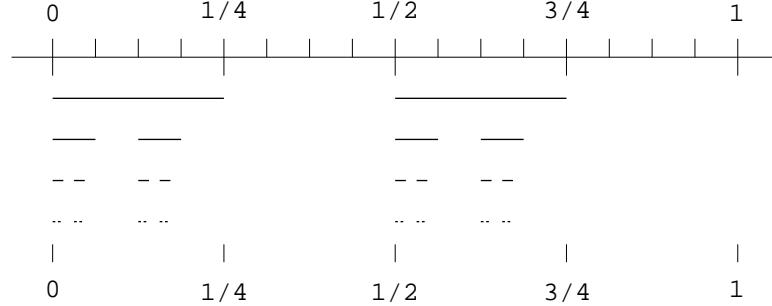


Fig. 2. The Cantor set X_4

from always omitting the two choices 1 and 3. Equivalently (see Fig. 2), by repeated subdivision of $[0, 1]$, our Cantor set X results when, at each quarter subdivision step, we are omitting the second and the fourth open subintervals.

It is evident that $X \subset [0, 1]$ and that $\text{supp}(\mu) = X$ where $\text{supp}(\mu)$ denotes the support of μ . Furthermore, the measure μ is the restriction of the Hausdorff measure $\text{Haus}_{\frac{1}{2}}$ of Hausdorff dimension $\frac{1}{2}$. We shall be interested in the algorithms for ONBs in the Hilbert space $L^2(\mu) = L^2(X_4, \mu_4)$. One reason for the choice of this particular example is that it appears to be prototypical of IFSs for which $L^2(\mu)$ has an ONB consisting of complex exponentials $e_\lambda(x) := e^{i2\pi\lambda x}$. Specifically, the co-authors of [JoPe98] prove that $\{e_\lambda \mid \lambda \in \{0, 1, 4, 5, 16, 17, 20, 21, \dots\}\}$ is an ONB in $L^2(\mu)$. (Note that the index set for this ONB is the set of all finite sums

$$\left\{ k_0 + k_1 4 + k_2 4^2 + \dots + k_s 4^s \mid k_i \in \{0, 1\} \right\}.$$

There are other choices of ONBs, see, e.g., [DuJo05] and [DuJo06a], but the argument for orthogonality is based on a form of factorization of the following transform:

$$\begin{aligned} \hat{\mu}(\lambda) &:= \int e_\lambda(x) d\mu(x) \\ &= \frac{1}{2} (1 + e^{i\pi\lambda}) \hat{\mu}\left(\frac{\lambda}{4}\right) \\ &= \prod_{m=0}^{\infty} \frac{1}{2} (1 + e^{i\pi\lambda 4^{-m}}), \quad \lambda \in \mathbb{R}. \end{aligned}$$

The two operators S_0, S_1 generating our associated representation of \mathcal{O}_2 in $L^2(X_4, \mu_4)$ are specified as follows:

$$(S_0 f)(x) = \begin{cases} f(4x), & x \in \tau_0(X_4), \\ f(4x - 2), & x \in \tau_1(X_4), \end{cases} \quad (4.3)$$

and

$$(S_1 f)(x) = \begin{cases} f(4x), & x \in \tau_0(X_4), \\ -f(4x - 2), & x \in \tau_1(X_4). \end{cases} \quad (4.4)$$

We have $S_0 e_0 = e_0$, and the formula (4.1) takes the form

$$\int_X f \, d\mu = \int_X (S_0^* f)(x) \, d\mu(x), \quad f \in C(X).$$

We now prove that the representation of \mathcal{O}_2 outlined above for the Cantor example is *irreducible*.

Theorem 4.1. *Let $\mathcal{H} = L^2(X_4, \mu_4)$ be the Hilbert space defined from the Cantor measure μ_4 with $\text{supp}(\mu_4) = X_4$. Let S_i , $i = 0, 1$, be the operators defined in (4.3) and (4.4) above.*

These two operators then generate an irreducible representation of the Cuntz C^ -algebra \mathcal{O}_2 acting on the Hilbert space \mathcal{H} .*

PROOF. We first check that the two operators S_i , $i = 0, 1$, are isometries. The calculation uses the three facts:

- (i) $\tau_0(X) \cap \tau_1(X) = \emptyset$,
- (ii) $\tau_0(X) \cup \tau_1(X) = X$,
- (iii) $\mu(cE) = \sqrt{c}\mu(E)$ for all $c \in \mathbb{R}_+$ and all $E \in \mathcal{B}_{\frac{1}{2}}(X)$ = the sigma-algebra of the Haus $_{\frac{1}{2}}$ -measurable sets.

Now if f is bounded and measurable, then

$$\begin{aligned} \int_X |S_i f|^2 \, d\mu &= \int_{\tau_0(X)} |f(4x)|^2 \, d\mu(x) + \int_{\tau_1(X)} |f(4x-2)|^2 \, d\mu(x) \\ &= \int_X |f(y)|^2 \, d\mu(\tau_0(y)) + \int_X |f(y)|^2 \, d\mu(\tau_1(y)) \\ &= \frac{1}{2} \int_X |f|^2 \, d\mu + \frac{1}{2} \int_X |f|^2 \, d\mu = \int_X |f|^2 \, d\mu. \end{aligned}$$

The remaining Cuntz property, $S_0 S_0^* + S_1 S_1^* = \mathbb{1}_{\mathcal{H}}$, is immediate from

$$\mu = \frac{1}{2} (\mu \circ \tau_0^{-1} + \mu \circ \tau_1^{-1}).$$

Set $e_0 = \varphi = \chi_X$ = the indicator function of the Cantor set $X = X_4$. We claim that the vectors $S_J \varphi$, $J \in \mathcal{M}$ (= all finite 0–1 multi-indices), span a dense subspace in \mathcal{H} . Since $S_0^* \varphi = \varphi$ and $S_1^* \varphi = 0$, it follows that φ is then a cyclic vector for the \mathcal{O}_2 -representation; and that $\omega_\varphi := \langle \varphi | \cdot \varphi \rangle$ is a Cuntz state. An application of [BJP96, Theorem 3.3] then yields irreducibility.

For all multi-indices J, K of the same length, we set $JK := \sum_i j_i k_i$. Recall

$j_i, k_i \in \{0, 1\}$ for \mathcal{O}_2 . An induction now yields

$$\chi_{\tau_J(X)} = \frac{1}{|J|} \sum_{\substack{K, \\ |K|=|J|}} (-1)^{JK} S_K \varphi. \quad (4.5)$$

But by the Cantor construction, we know that these indicator functions

$$\left\{ \chi_{\tau_J(X)} \mid J \in \mathcal{M} \right\} \quad (4.6)$$

span a dense subspace in $\mathcal{H} = L^2(\mu_4)$. The fact that the functions in (4.6) are total in $L^2(\mu_4)$ is a consequence of the recursive algorithm used in the construction of the measure μ_4 (see Fig. 2). Moreover, the argument for the present special case in fact carries over *mutatis mutandis* to general IFS constructions, regardless of whether the limit measure μ is fractal or not; see Section 2 above and [Hut81] for further details.

This concludes the proof of our theorem. \square

Corollary 4.2. [JoPe98] *Consider the Cantor system $(X, \mu) = (X_4, \mu_4)$, the Hilbert space $\mathcal{H} = L^2(X, \mu)$, and the functions*

$$\begin{aligned} e_n(x) &:= e^{i2\pi nx}, \\ n &\in \{0, 1, 4, 5, 16, 17, 20, 21, \dots\} \\ &= \left\{ j_0 + j_1 4 + j_2 4^2 + \dots + j_p 4^p \mid j_i \in \{0, 1\} \right\} =: \Lambda. \end{aligned} \quad (4.7)$$

These functions form an ONB in \mathcal{H} , and

$$\mathcal{H} = \mathbb{C} \varphi \oplus \sum_{\substack{m: \\ m \text{ odd,} \\ m \in \Lambda}}^{\oplus} \mathcal{H}(e_m). \quad (4.8)$$

PROOF. The result is a direct corollary of the two theorems, Theorem 3.1 and Theorem 4.1. In the present application it turns out that $K(e_m) = 0$ for all m odd in Λ . As a result we get that $\mathcal{H}(e_m)$ is spanned by $S_0^k e_m$, $k \in \mathbb{N}_0$; and recall that $S_0^k e_m = e_{m4^k}$, m odd in Λ . \square

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